Event collisions in systems with delayed switches

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We study dynamical systems that switch between two different vector fields depending on a discrete variable. If the two vector fields have linearly unstable fixed points and the switch is subject to a hysteresis and to a delay one expects the system to switch periodically back and forth between the two vector fields, always switching at certain submanifolds of the state space. This is true as long as the delay is sufficiently small. When the delay reaches a problem-dependent critical value so-called event collisions can occur. We show that at these event collisions the switching manifolds can increase their dimension, giving rise to higher-dimensional dynamics near the periodic orbit than expected. In many practical applications such as control engineering the dynamical system has additional symmetry, which adds difficulty in the analysis because event collisions can occur at several points along the periodic orbit simultaneously.

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1. Introduction

The motivation behind this paper is the observation that hybrid dynamical systems arising in practical applications often show a surprisingly intricate dynamical behavior if their switch is subject to a delay [1, 2, 3]. The observed behavior often does not match the simple classification of possible codimension-one bifurcations as given in [4]. The reason behind this phenomenon is that practical systems often have special symmetries (for example piecewise affine systems as arising in control engineering have a full reflection symmetry). The presence of this symmetry causes a violates of the genericity assumptions made in [4]. Most prominently, periodic orbits of systems with delayed switching typically have 'corners'. A classical event collision corresponds to the case when, varying a parameter, one of the corners of the orbits crosses a switching boundary [4]. In the examples of [1, 2, 3] the symmetry enforces that one of the other corners of the periodic orbit *simultaneously* crosses a switching boundary (at least for symmetric periodic orbits), which violates the assumptions of [4]. This paper studys the simplest but most common case of symmetric hybrid systems, namely systems with full reflection symmetry. This symmetry is present in all examples studied in [1, 2, 3].

Let us consider hybrid dynamical systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t-\tau)) \\ u(t) &= \begin{cases} -1 & \text{if } h(x) \geq 1, \text{ or,} \quad \text{if } h(x) \in (-1,1) \text{ and } u_-(t) = -1 \\ 1 & \text{if } h(x) \leq -1, \text{ or, if } h(x) \in (-1,1) \text{ and } u_-(t) = 1 \end{cases} \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \{-1,1\}$, $f : \mathbb{R}^n \times \{-1,1\} \mapsto \mathbb{R}^n$ is smooth and $h : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous and piecewise smooth with a nonzero gradient $H(x) = \partial_x h(x)$. In the definition of $u(t)$ in (1), $u_-(t)$ is defined by

$$u_-(t) := \lim_{s \nearrow t} u(s),$$

which gives rise to a switch between $u = -1$ and $u = +1$ with hysteresis. In control problems, typically, x is the state variable, and u is a discrete control input governed by the switching law in (1). For a trajectory $(x(\cdot), u(\cdot))$ of (1) we call t a *crossing time* if $u(t)u_-(t) = -1$ (that is, $u(t) \neq u_-(t)$). The level sets $(-\infty, -1]$, $(-1,1)$ and $[1, \infty)$ of the function h provide a partition of $\mathbb{R}^n$ into disjoint sets with piecewise smooth boundaries given by $\{h(x) = \pm 1\}$. Thus, at a crossing time $x(t)$ crosses either $\{h(x) = 1\}$ from $\{h(x) < 1\}$ to $\{h(x) \geq 1\}$, or $\{h(x) = -1\}$ from $\{h(x) > -1\}$ to $\{h(x) \leq -1\}$. The state variable x in system (1) follows at any instance of time one of the two flows φ_+ or φ_- where $\varphi_\pm$ is generated by the ordinary differential equation $\dot{x} = f(x, \pm 1)$. That is, $x(t+\delta) = \varphi_+^\delta x(t)$ if $u([t-\tau, t-\tau+\delta]) = 1$, and $x(t+\delta) = \varphi_-^\delta x(t)$ if $u([t-\tau, t-\tau+\delta]) = -1$.

A special feature of system (1) is that the switch introduced by u is subject to a delay. That is, the evolution of x depends, via u, on the value of x from time τ ago. Typically, systems with delayed switches admit non-stationary, periodic or more complicated, dynamics [5, 6, 3]. The phase space of system (1) is infinite-dimensional because it is necessary to keep track of the history of u in $[t-\tau, t]$ to determine the forward evolution of x and u. More precisely, an appropriate initial value for (1) would be a tuple (x_0, u_0) consisting of $x_0 \in \mathbb{R}^n$, and a function $u_0 : [-\tau_{\max}, 0) \mapsto \{-1,1\}$ of, say, bounded variation (that is, $u \in BV := BV([-\tau_{\max}, 0]; \{-1,1\})$ where $\tau_{\max}$ is an upper bound for the delay). However, it is well known that the dynamics close to a periodic orbit $L = (x(\cdot), u(\cdot))$ of (1) can be described locally by a finite-dimensional smooth map M if L satisfies the following two genericity conditions.

Condition 1 (Generic periodic orbits) *All crossing times t of L satisfy the following two conditions.*

1. (Absence of event collision) *The time $t - \tau$ is a not a crossing time of L.*

2. (Transversality) *The gradient H of h is well defined and continuous in $x(t)$ and satisfies $H(x(t))\dot{x}(t) \neq 0$.*

Condition 1.1 automatically guarantees that $x(\cdot)$ is continuously differentiable in all crossing times, making th time derivative of x in Condition 1.2 well defined. If the orbit L is also *slowly oscillating*, that is, if the time differences between subsequent crossing times are all larger than the delay τ, then the map M is $n-1$-dimensional. In this case the map M can be obtained by recording the first return map (Poincaré map) to a local cross-section (Poincaré section) in $\mathbb{R}^n$, transversal to the graph of $x(\cdot)$ in a point $x(t)$ where u is constant in $[t - \tau, t]$. A reduction of the description of the dynamics of (1) near L to the smooth finite-dimensional map M links the bifurcation theory of slowly oscillating periodic orbits satisfying Condition 1 to the classical bifurcation theory of smooth finite-dimensional maps [7]. Reference [4] proves that this reduction to (higher-dimensional) smooth maps works also for periodic orbits that are not slowly oscillating as long as Condition 1 is satisfied. Furthermore, reference [4] classifies what can happen generically near slowly oscillating periodic orbits that violate one of the conditions 1.1 or 1.2. It derives that the local return maps are piecewise smooth $n-1$-dimensional maps assuming certain secondary genericity conditions. This links the theory of codimension-one discontinuity-induced bifurcations of slowly oscillating periodic orbits to the theory of finite-dimensional piecewise smooth maps [8, 9, 10, 11].

The motivation behind this paper is the observation that many systems arising in applications have special symmetry properties that obstruct the application of the generic theory of [4]. The symmetry of the periodic orbit often implies that a collision (that is, the violation of Condition 1.1) for one crossing time t leads automatically to a simultaneous collision for all other crossing times, which violates the secondary conditions assumed in [4]. This has been observed in the example system studied extensively in [1, 2] as well as in many of the examples discussed in [3]. The major source of examples of systems of the form (1) is control engineering where often f is linear and the switching law is affine, that is,

$$\begin{aligned} f(x,u) &= Ax + bu \\ h(x) &= h^T x \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, and $b, h \in \mathbb{R}^n$. The form (2) implies that system (1) has the $\mathbb{Z}_2$ symmetry of full reflection at the origin $(x, u) \mapsto (-x, -u)$, which occurs if the right-hand-side of (1) satisfies

$$\begin{aligned} f(x,u) &= -f(-x,-u) \\ h(x) &= -h(-x). \end{aligned} \tag{3}$$

The $\mathbb{Z}_2$ reflection symmetry (3) typically gives rise to a symmetric periodic orbit $L = (x(\cdot), u(\cdot))$ satisfying $x(t-T) = -x(t)$ and $u(t-T) = -u(t)$ for the half-period T and all t. An event collision of this type symmetric periodic orbit for a crossing time t automatically induces a simultaneous event collision for the crossing time $t - T$, a scenario which is not covered by the classification of [4]. We point out that affine systems of the form (1)–(2) can exhibit complex behavior, including chaos, even though all ingredients of the right-hand-side are linear. The switch governing u in (1) is a strong nonlinearity that is a common cause of complicated dynamics.

2. Local return maps of symmetric periodic orbits at event collisions

Let us suppose that system (1), (3) has a symmetric periodic orbit $L_* = (x_*(\cdot), u_*(\cdot))$ of half-period T which, for a critical delay τ_*, experiences an event collision for crossing time 0, and, enforced by the reflection symmetry, for crossing time T. For compactness of presentation let us assume that 0 and T are the only crossing times of L_*. Thus, $T = \tau_*$, and x_* switches between the flows φ_+ and φ_- at the crossing times 0 and τ_*. Consequently, (without loss of generality) L_* consists of the two segments

$$\begin{aligned} x_*([0,\tau_*]) &= \varphi_+^{[0,\tau_*]}(x_*(0)), & u_*([0,\tau_*)) &= -1, \\ x_*([\tau_*,2\tau_*]) &= \varphi_-^{[0,\tau_*]}(-x_*(0)) = -x_*([0,\tau_*]), & u_*([\tau_*,2\tau_*)) &= 1. \end{aligned}$$

Moreover, $h(x_*(0)) = 1$ and $h(x_*(\tau_*)) = -1$. The following transversality condition guarantees that the evolution of system (1), (3) is continuous in L_*:

Condition 2 (Continuous event collision of symmetric orbits) *The orbit L_* intersects the switching manifold $\{x : h(x) = 1\}$ transversally at time 0:*

$$q := H(x_*(0))f_+ \cdot H(x_*(0))f_- > 0$$

where $f_+ = f(x_*(0), 1)$ and $f_- = f(x_*(0), -1)$.

Condition 2 means that, even though the orbit $x_*(\cdot)$ is not differentiable in its crossing times 0 and τ_*, it still crosses the switching manifolds $\{x : h(x) = \pm 1\}$ transversally in the sense that the left- and the right-sided time derivatives of $x_*(\cdot)$ both point through the switching manifold and both point in the in the same direction. Condition 2 is formulated for crossing time 0. The reflection symmetry implies that the same condition automatically holds also for the crossing time τ_*.

Consider the following set of initial conditions in the vicinity of L_* (choosing $\Delta < \tau$ such that $\tau + \Delta \le \tau_{\max}$ for all $\tau \approx \tau_*$ and denoting a sufficiently small neighborhood of a point $x \in \mathbb{R}^n$ by $U(x)$):

$$\begin{aligned} U_\tau := \{(x, u(\cdot)) \in \mathbb{R}^n \times BV : u((-\tau - \Delta, -\tau)) = -1,& \\ u([-\tau, -\tau + \Delta)) = 1, x \in U(x_*(0))\}&. \end{aligned}$$

The set U_τ is the set of initial conditions that switch exactly once from φ_- to φ_+ in $U(x_*(0))$. The periodic orbit L_* is an element of U_{τ_*}. For a given delay τ the set U_τ can be identified with $U(x_*(0))$, thus, defining a topology on U_τ. Condition 2 guarantees that for delays τ close to τ_* the set U_τ is invariant relative to a sufficiently small neighborhood $\tilde{U}$ of L_*. The x-components of all trajectories starting from elements of U_τ switch exactly once from φ_- to φ_+ in $U(x_*(0))$, then follow φ_+ to $U(x_*(\tau_*)) = U(-x_*(0))$, then switch from φ_+ to φ_- in $U(x_*(\tau_*))$, and then return to $U(x_*(0))$, following φ_- (as long as the trajectory does not leave $\tilde{U}$). Consequently, for $\tau = \tau_* + \delta \approx \tau_*$ the system (1), (3) restricted to U_τ defines a return map $M : D(M) \subset U_\tau \mapsto U_\tau$ where $D(M)$, the domain of definition of M, is an open subset of U_τ. The following theorem states that M describes the behavior of system (1), (3) for small δ and near L_* completely. Moreover, it provides a formula for the piecewise smooth map M.

Theorem 3 (Dynamics near symmetric collisions) *Let τ be sufficiently close to τ_* and let $\tilde{U}$ be a sufficiently small neighborhood of L_*. Then all initial conditions in $\tilde{U}$ are mapped into $D(M)$ by* *system* (1), (3) *within a finite time less than* $3\tau_*$. *On* $D(M)$ *the map* M *is given as* $M = F_\tau \circ F_\tau$ *where* $F\tau : U(x_*(0)) \mapsto U(x_*(0))$ *is defined as*

$$F_\tau(x) = -\varphi_+^{\tau+t(x)} x \tag{4}$$

and $t(x) \in (-\tau, \tau)$ *is the unique time such that*

$$\begin{cases} h\left(\varphi_+^{t(x)} x\right) = 1 & \textit{if } h(x) \leq 1, \\ h\left(\varphi_-^{t(x)} x\right) = 1 & \textit{if } h(x) > 1. \end{cases} \tag{5}$$

The definition of the (negated) half-return map F_τ identifies elements of U_τ with their x-components, which is justified as explained above. The definition of the traveling time $t(x)$ implies that F is continuous in $D(M)$ and smooth in its two subdomains $D_- := U(x_*(0)) \cap \{x : H(x) \leq 1\}$ and $D_+\tau) := U(x_*(0)) \cap \{x : H(x) > 1\}$ but, in general, its derivative has a discontinuity along the boundary D_0 between D_- and D_+. The existence and uniqueness of the traveling time is a consequence of the transversality Condition 2. The linearizations of both parts of F_τ with respect to x and τ in $x_*(0)$ and τ_* are:

$$F_{\tau_*+\delta}(x_*(0) + \xi) - x_*(0) = -A^{\tau_*} \left[\left[I - \frac{f_+ H_*}{g} \right] \xi + \delta f_+ \right] + O(|(\xi, \delta)^2|) \tag{6}$$

where $H_* = H(x_*(0))$, $A^{\tau_*} = \partial_x \left[\varphi_+^{[0,\tau_*]} x \right]_{x = x_*(0)}$, $f_+ = f(x_*(0), 1)$, and

$$g = \begin{cases} H_* f_+ & \text{if } x_*(0) + \xi \in D_- \setminus D_0, \\ -H_* A^{\tau_*} f_+ & \text{if } x_*(0) + \xi \in D_+. \end{cases} \tag{7}$$

Condition 2 asserts that the product q of $H_* f_+$ and $H_* f_- = -H_* A^{\tau_*} f_+$ is nonzero. Thus, Condition 2 implies that g in nonzero in both cases. The affine approximation of the boundary D_0 between D_- and D_+ in $x_*(0)$ is given by $\{x_*(0) + \xi : H_* \xi = 0\}$. The map F_τ projects the whole subdomain D_- onto the $n-1$-dimensional local submanifold $\{x \in U(x_*(0)) : h(-\varphi_+^{-\tau} x) = 1\}$ which is the delayed switching manifold. Correspondingly, its linearization (6) projects ξ linearly by $I - f_+ H_* / (H_* f_+)$ before propagating it by $-A^{\tau_*}$. Consequently, an event collision for a symmetric periodic orbit satisfying Condition 2 increases the dimension of the image of the local return map from $n-1$ in D_- to n in D_+.

3. Illustrative example

This section will illustrate the most common scenario for the dynamics near an event collision of a symmetric periodic orbit. We will use a two-dimensional example of the piecewise linear form (1)–(2) to demonstrate how the increase of the dimension of the return map of the periodic orbit manifests itself near an event collision.

We choose $h^T = (1, 0)$ (without loss of generality) and the right-hand-side parameters A and b in (2) such that, for a critical delay $\tau = \tau_*$ system (1)–(2) has a symmetric periodic orbit $L_* = (x_*(\cdot), u_*(\cdot))$ with half-period τ_* and crossing times 0 and τ_*. Furthermore, we choose A, b and τ_* in a manner such that A, b and L_* satisfy the following conditions.

1. The matrix A and the critical delay τ_* satisfy $A^{\tau_*} := \exp(A\tau_*) = r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with $r > 1$.

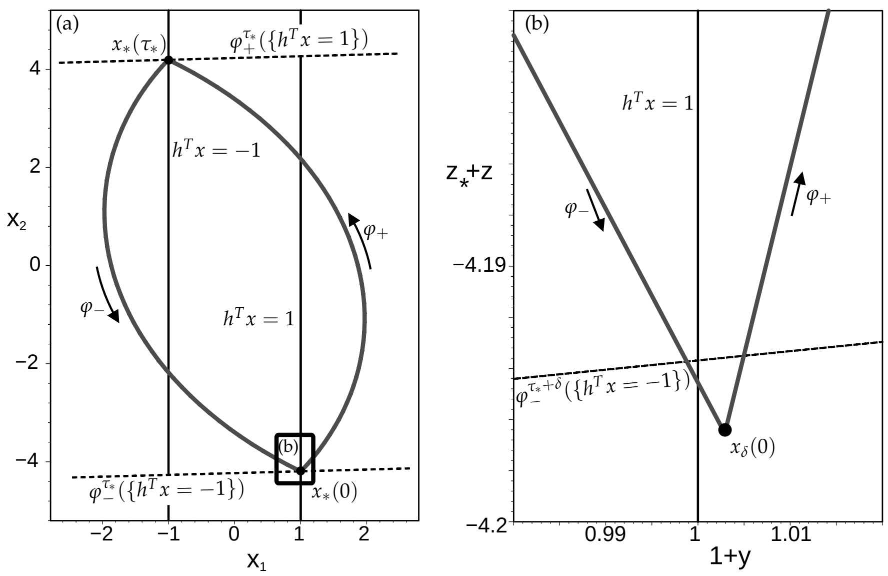

Figure 1: Phase portrait of a colliding periodic orbit as discussed in Section 3. The dashed lines are the time τ_* images of the lines $\{h^T x = y = \pm 1\}$ under the flows φ_+ and φ_-, respectively. The periodic orbit L_* (in (a)) switches at times 0 and τ_* due to the crossing of the line $\{y = \pm 1\}$ time τ_* ago simultaneous to its crossing of the other switching line $\{y = \mp 1\}$. How this collision is unfolded by increasing the delay to $\tau_* + \delta$ is shown in the zoom (b). The delay $\tau_* + \delta$ is larger than the half-period and the periodic orbit no longer switches at the delayed switching lines.

2. The colliding orbit x_* intersects the switching line $\{x : h^T x = 1\}$ in $x_*(0) = (1, z_*)^T$ and the switching line $\{x : h^T x = -1\}$ in $x_*(\tau_*) = (-1, -z_*)^T$ transversally. That is, $h^T f_+ > 1$ and $h^T f_- > 0$ where

$$f_+ := \frac{d}{dt}\varphi_+^t|_{t=0} x_*(0) = A x_*(0) + b$$
$$f_- := \frac{d}{dt}\varphi_-^t|_{t=0} x_*(0) = A x_*(0) - b = -A^{\tau_*} f_+.$$

   The coordinate z_* is uniquely defined by A and τ_* (as required in point (1) and f_+. The concrete expression for z_* is given in the Appendix A.

3. The vector f_+ has the form $f_+ = (1, c)^T$, which implies that $f_- = (rc, -1)$ due to point (1), and $c > 0$ due to point (2).

Thus, both crossing times of L_* violate genericity condition 1.1 simultaneously. Figure 1(a) shows a phase portrait of L_* (grey, thick, solid line), superimposed with the non-delayed switching lines $\{h^T x = \pm 1\}$ (solid) and their time τ_* images $\varphi_\pm^{\tau_*}\{h^T x = \pm 1\}$ (dashed).

Condition (1) implies that A must have a complex pair of eigenvalues with positive real part and that τ_* is such that the rotation angle α induced by $\exp(A\tau_*)$ is $\pi/2$ (which results in the simple form of $\exp(A\tau_*)$ in point (1)). Condition (2) corresponds to the transversality Condition 2 of Section 2. If condition (2) is satisfied the choice $f_+ = (1, c)$ can be made without loss of generality. Appendix A describes in detail how the parameters r, c and α uniquely define A, b, the critical delay τ_* and the colliding symmetric periodic orbit $x_*(\cdot)$ of period $2\tau_*$. The requirement that the delay τ_* is non-negative and the condition $h^T f_- > 0$ (part of condition 2) restrict the set of admissible r and c by the conditions $r > 1$, $c > 0$, and

$$c > \frac{r\pi - 2\log r}{\pi + 2r\log r} \tag{8}$$

(see Appendix A).

The linearizations (6), (7) (where $H_* = (1, 0)$) approximate the square root F_τ of the local return map (which is nonlinear, even though system (1)–(2) is piecewise affine) for $x = [1, z_*]^T + [y, z]^T$ and $\tau = \tau_* + \delta$ for small $[y, z]^T$ and δ. Expressed in the quantities r, c, $[y, z]$ and δ the linearized F_τ reads (truncated to affine terms)

$$F_{\text{lin}} : \begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} -rcy + rz + rc\delta \\ -r\delta \end{bmatrix} & \text{if } y \le 0 \\ \begin{bmatrix} -y + rz + rc\delta \\ (c^{-1} - r)y - r\delta \end{bmatrix} & \text{if } y > 0. \end{cases} \tag{9}$$

Hence, if $c > r$ (then also restriction (8) is satisfied) the map F_{lin} has the fixed point

$$\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{cases} \begin{bmatrix} r\delta(c-r)/(rc+1) \\ -r\delta \end{bmatrix} & \text{if } \delta < 0, \\ \begin{bmatrix} r\delta c(c-r)/(2c + r^2c - r) \\ -r\delta c(rc+1)/(2c + r^2c - r) \end{bmatrix} & \text{if } \delta > 0, \end{cases}$$

Thus, there exists a continuous family $x_\delta = [y_\delta, z_\delta]^T$ of fixed points of F_{lin} for $\delta \in (-\delta_0, \delta_0)$. This continuous family persists under the small perturbation toward F_τ. The fixed points in F_τ correspond to a family of symmetric periodic orbits of system (1)–(2) which has an event collision at $\delta = 0$. Apparently, the fixed point x_δ has only one nonzero eigenvalue (corresponding to a nonzero Floquet multiplier of the periodic orbit) for $\delta < 0$. This eigenvalue is $rc > 1$. Hence, x_δ is unstable with only one nontrivial direction.

When $\delta > 0$, the fixed point x_δ has a complex conjugate pair of eigenvalues $-1/2 \pm \sqrt{1/4 - r^2 + r/c}$ if $c > (r - 1/(4r))^{-1}$ which is stable if $c < (r - 1/r)^{-1}$, loosing its stability in a 1 : 3 resonant torus bifurcation at $c = (r - 1/r)^{-1}$. Consequently, for $r \in (\sqrt{5}/2, \sqrt{2})$ and for all $c \in (r, (r - 1/r)^{-1})$ there exists a stable symmetric periodic orbit of system (1)–(2) with a stable complex conjugate pair of Floquet multipliers for all $\tau = \tau_* + \delta$ and $\delta > 0$ sufficiently small. The degeneracy of the strong (1 : 3) resonance is caused by our selection of the rotation angle $\alpha = \pi/2$ in the choice of the parameters in A and τ_*. In general, no torus will emanate from this resonant torus bifurcation [7]. Variation of the angle α will unfold this degeneracy.

Figure 2 shows the iterations of the nonlinear map F_τ (the exact negated half-return map of system (1)–(2)) for $\tau = \tau_* + \delta$ where $\delta = 0.1$, $c = 3/2$ and $r = (1 + \sqrt{10})/3 \approx 1.39$ (Fig. 2(a)) and $r = \sqrt{2}$ (Fig. 2(b)). Fig. 2(a) gives evidence of a stable fixed point with two complex conjugate eigenvalues in approximate 1 : 3 resonance whereas Fig. 2(b) clearly shows an unstable fixed point at the center (and, possibly, the period three saddle-type orbit which is generically present near 1 : 3 resonances). Fig. 1(b) shows a zoom-in into the neighborhood of $x_*(0)$ of the phase portrait of the periodic orbit L_δ where $\delta = 0.1$ corresponding to the stable fixed point in Fig. 2(a). Clearly, the periodic orbit L_δ does not switch at the delayed

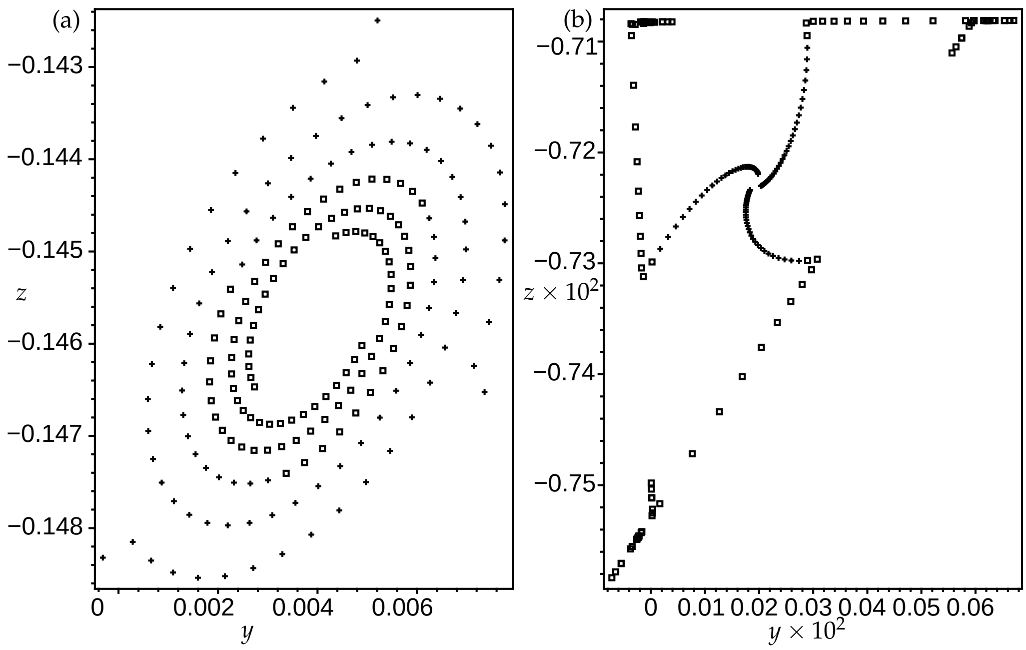

Figure 2: Iterations of the half-return map F_τ starting near the fixed point. The first 100 iterates are marked by crosses, the iterates 101–200 are marked by boxes, giving evidence of a stable fixed point for $c = 3/2$, $r = (1+\sqrt{10})/3 \approx 1.39$, $\delta = 0.1$ ($\tau = \tau_* + \delta$) and an unstable fixed point for $c = 3/2$, $r = \sqrt{2}$.

switching line $\varphi_-^{\tau_*+\delta}\{h^T x = -1\}$. The half-period of L_δ is less than $\tau_* = \delta$. Thus, L_δ is a stable symmetric periodic orbit that is not slowly but *rapidly* oscillating. Hence, this example also illustrates that a stable symmetric periodic orbit can be created in an event collision by increasing the delay beyond the critical value τ_*.

4. Conclusion

The paper discusses the dynamics near periodic orbits in hybrid dynamical systems. It is motivated by the fact that in many practical applications the presence of symmetry prevents the generic and simple bifurcation scenarios as classified in [4] but instead gives rise to intricate and counterintuitive event collision phenomena [3, 1, 2].

We describe and unfold the simplest and most common case of an event collision in a symmetric system as it occurs for example in a piecewise affine system which is switching with hysteresis and delay. In this case two corners of a symmetric periodic orbit simultaneously collide with a switching manifold. This causes an increase of the dimension of the phase space for the return map along the periodic orbit. We demonstrated this fact with a simple two-dimensional piecewise linear example which has a symmetric periodic orbit with a weakly stable or unstable complex conjugate pair of eigenvalues if the delay is greater than the critical value corresponding to an event collision.

The analysis of the possible dynamics in the unfolding of the event collision is far from complete but the initial theoretical results presented in this paper will lead to classifications of practically relevant behavior for concrete systems.

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A. Reconstruction of the right-hand-side

This appendix describes in which sense the right-hand-side parameters A, b and τ_* of the piecewise affine system (1)–(2) are uniquely determined by the artificial parameters c, r and α which we use in Section 3 to describe the return map of the colliding periodic orbit L_*.

Let us denote the first component of the state variable $x \in \mathbb{R}^2$ by y and the second component by z. We choose $h = (1, 0)^T$ and the matrix $A = \tau_*^{-1} A_0$ where

$$A_0 = \begin{bmatrix} \log r & -\alpha \\ \alpha & \log r \end{bmatrix} \tag{10}$$

and $r > 1$. Thus, the flows $\varphi_\pm$ are given by

$$\varphi_\pm^t x = \exp(At)x \pm (\exp(At) - I)\tau_* A_0^{-1} b,$$

spiraling outward from the unstable sources $\mp\tau_* A_0^{-1} b$. Consequently, system (1)–(2) with delay τ_* has a symmetric periodic orbit that switches from φ_- to φ_+ in $x_*(0) = (1, z_*)^T$ on the switching line $h^T x = 1$ if

$$-\begin{bmatrix} 1 \\ z_* \end{bmatrix} = \exp A_0 \begin{bmatrix} 1 \\ z_* \end{bmatrix} + (\exp A_0 - I)\tau_* A_0^{-1} b.$$

Furthermore, if we prescribe the time derivative of the outgoing flow φ_+ in $(1, z_*)^T$ by $(1, c)$ then the parameters A, b and τ_* and the periodic orbit are uniquely determined by the parameters r, α and c and the relations

$$\begin{aligned} \begin{bmatrix} \tau_*^{-1} \\ \tau_*^{-1} z_* \end{bmatrix} &= \frac{1}{2}\left[I - \exp A_0\right] A_0^{-1} \begin{bmatrix} 1 \\ c \end{bmatrix}, \qquad b = \left[\exp A_0 + I\right] \begin{bmatrix} 1 \\ c \end{bmatrix} \\ x_*(\pm t) &= \exp(\pm A_0 \tau_*^{-1} t) \begin{bmatrix} 1 \\ z_* \end{bmatrix} \pm \left[\exp(\pm A_0 \tau_*^{-1} t) - I\right] \tau_* A_0^{-1} b \end{aligned} \tag{11}$$

where A_0 is given by (10) and t runs from 0 to 1. Hence, the relations (11) allow one to study the return map near the colliding periodic orbit in dependence of the parameters r, c and α. We choose $r > 1$ and, for convenience, $\alpha = \pi/2$ in Section 3, implying that $\exp A_0 = \left[\begin{smallmatrix} 0 & -r \\ r & 0 \end{smallmatrix}\right]$. This results in the relation

$$\tau_* = \frac{4\log^2 r + \pi^2}{2\log r - r\pi + c\pi + 2cr\log r}.$$

for τ_*. The condition $\tau_* \geq 0$ implies the admissibility condition on c

$$c > \frac{r\pi - 2\log r}{\pi + 2r\log r},$$

which is always satisfied if $c > r > 1$.